Lecture-8

Solution of Linear System

Solution of A linear System

- Gaussian Elimination, Backward Substitution
- Matrix Factorization
- Iterative Techniques

$$Ax = b$$

$$\sum_{j=1}^{n} (a_{ij} x_j) = b_i \quad \text{for } i = 1, 2, \ldots, n$$

Gaussian Elimination, Backward Substitution

6.1 **Gaussian Elimination with Backward Substitution**

To solve the $n \times n$ linear system

$$
\begin{aligned}
E_1 &: \; a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\
E_2 &: \; a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\
&\;\;\vdots \\
E_n &: \; a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n,n+1}
\end{aligned}
$$

INPUT number of unknowns and equations n; augmented matrix $A =$
$i \le n$ and $1 \le j \le n+1$.

OUTPUT solution $x_1, x_2, \ldots, x_n$ or message that the linear system has no

Step 1 For $i = 1, \ldots, n-1$ do Steps 2–6. (*Elimination process.*)

Step 2 Let p be the smallest integer with $i \le p \le n$ and $a_{pi} \ne 0$.
If no integer p can be found
then OUTPUT ('no unique solution exists');
STOP.

Step 3 If $p \ne i$ then perform $(E_p) \leftrightarrow (E_i)$.

Step 4 For $j = i+1, \ldots, n$ do Steps 5 and 6.

Step 5 Set $m_{ji} = a_{ji}/a_{ii}$.

Step 6 Perform $(E_j - m_{ji}E_i) \to (E_j)$;

Step 7 If $a_{nn} = 0$ then OUTPUT ('no unique solution exists');
STOP.

Step 8 Set $x_n = a_{n,n+1}/a_{nn}$. (*Start backward substitution.*)

Step 9 For $i = n-1, \ldots, 1$ set $x_i = \left[a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right] \Big/ a_{ii}$.

Step 10 OUTPUT $(x_1, \ldots, x_n)$; (*Procedure completed successfully.*)
STOP.

Iterative Methods for Solving Linear Systems

- For large sparse system Gaussian Elimination and Backward substitution is not suitable.
- Approximate solution using iterative methods

Jacobi

$$\sum_{j=1}^{n} (a_{ij} x_j) = b_i \quad \text{for } i = 1,2,\dots,n$$

$$x_i^k = \frac{\sum_{j=1,\, j\neq i}^{n} (-a_{ij} x_j^{k-1}) + b_i}{a_{ii}} \quad \text{for } i = 1,2,\dots,n$$

$$X = TX + C$$

Jacobi

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \dots & 0 \\ -a_{21} & & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -a_{n1} & \dots & -a_{nn-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \dots & -a_{1n} \\ 0 & & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{n-1n} \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$= D \qquad - \qquad L \qquad - \qquad U$$

Jacobi

$$AX = b$$

$$(D - L - U)X = b$$

$$DX = (L + U)X + b$$

$$X = D^{-1}(L + U)X + D^{-1}b$$

$$X^k = D^{-1}(L + U)X^{k-1} + D^{-1}b$$

$$X^k = TX^{k-1} + C$$

$$x_i^k = \frac{\sum_{j=1,\, j\neq i}^{n} (-a_{ij} x_j^{k-1}) + b_i}{a_{ii}} \quad \text{for } i = 1,2,\dots,n$$

Gauss-Seidel

$$\sum_{j=1}^{n} (a_{ij} x_j) = b_i \quad \text{for } i = 1,2,\dots,n$$

$$x_i^k = \frac{-\sum_{j=1}^{i-1} (a_{ij} x_j^k) - \sum_{j=i+1}^{n} (a_{ij} x_j^{k-1}) + b_i}{a_{ii}} \quad \text{for } i = 1,2,\dots,n$$

$$X^k = TX^{k-1} + C$$

Gauss-Seidel

$$AX = b$$

$$(D - L - U)X = b$$

$$(D - L)X^k = UX^{k-1} + b$$

$$X^k = (D - L)^{-1}UX^{k-1} + (D - L)^{-1}b$$

$$X^k = TX^{k-1} + C$$

Theorems

The sequence $x^k = Tx^{k-1} + c$ converges to the unique solution iff the spectral radius of T $r(T) < 1$.

If A is strictly diagonal dominant, then for any choice of x^0 both Jacobi and Gauss-Seidel methods give sequences that converge to a unique solution.

Interpretation of Gauss-Seidel

$$x_i^k = \frac{-\sum_{j=1}^{i-1}(a_{ij}x_j^k) - \sum_{j=i+1}^{n}(a_{ij}x_j^{k-1}) + b_i}{a_{ii}} \quad \text{for } i = 1,2,\ldots,n$$

$$r = b - A\tilde{x}$$

$$r_{ii} = b_i - \sum_{j=1}^{i-1}(a_{ij}x_j^k) - \sum_{j=i+1}^{n}(a_{ij}x_j^{k-1}) - a_{ii}x_i^{k-1}$$

$$r_{ii} + a_{ii}x_i^{k-1} = b_i - \sum_{j=1}^{i-1}(a_{ij}x_j^k) - \sum_{j=i+1}^{n}(a_{ij}x_j^{k-1})$$

$$r_{ii}^k + a_{ii}x_i^{k-1} = a_{ii}x_i^k$$

Interpretation of Gauss-Seidel

$$r_{ii}^k + a_{ii}x_i^{k-1} = a_{ii}x_i^k$$

$$x_i^k = x_i^{k-1} + \frac{r_{ii}^k}{a_{ii}}$$

$$x_i^k = x_i^{k-1} + w\frac{r_{ii}^k}{a_{ii}}$$

$$x_i^k = x_i^{k-1} + \frac{w}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n}(a_{ij}x_j^{k-1}) - a_{ii}x_i^{k-1}\right]$$

$$x_i^k = (1-w)x_i^{k-1} + \frac{w}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n}(a_{ij}x_j^{k-1})\right]$$

SOR (Successive Over Relaxation)

$$\sum_{j=1}^{n} (a_{ij} x_j) = b_i \quad \text{for } i = 1,2,\dots,n$$

$$x_i^k = (1-w)x_i^{k-1} + \frac{w}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^k - \sum_{j=i+1}^{n} (a_{ij} x_j^{k-1})\right]$$

$$w > 1$$

Theorem

If A is a PD and $0 < w < 2$ then SOR method converges for Any choice of initial approximation of solution x^0.

Theorem

If A is a PD and tri-diagonal, then

$$r(T_g) = r(T_j) < 1$$

Then optimal choice of w

$$w = \frac{2}{1 + \sqrt{1 - r(T_g)}}$$

$$w = \frac{2}{1 + \sqrt{1 - r(T_j)^2}}$$

Maximum eigen value